The Poisson Distribution

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1 Definition

The **Poisson distribution** with parameter $\lambda > 0$, denoted by $\mathrm{Poi}(\lambda)$, is a distribution over $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ such that the probability of any $k \in \mathbb{N}_0$ is

$$\mathrm{Poi}(\lambda)(k) = \frac{\lambda^k}{e^{\lambda} k!} \tag{1}$$

To remember this formula, first remember the Taylor series of e^x at $x = \lambda$ and divide both sides by e^{λ},

$$1 = \frac{1}{e^{\lambda}} + \frac{\lambda}{e^{\lambda}} + \frac{\lambda^2}{e^{\lambda} 2!} + \frac{\lambda^3}{e^{\lambda} 3!} + \cdots$$

Since the terms are positive and sum to 1, they form a valid distribution over $\mathbb{N}_0$.

1.1 Interpretation

$X \sim \mathrm{Poi}(\lambda)$ represents the number of heads in infinitely many independent random coin tosses where $\mathbf{E}[X] = \lambda$ (aka. rate). Formally, let $\mathrm{Bin}(n, p)$ denote the binomial distribution over $k \in \mathbb{N}_0$,

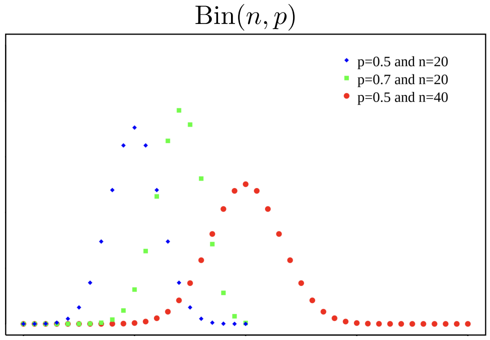
$$\mathrm{Bin}(n, p)(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Since $Z \sim \mathrm{Bin}(n, p)$ is equivalent to $Z = \sum_{i=1}^{n} Z_i$ where $Z_i \sim \mathrm{Ber}(p)$ independently, we have $\mathbf{E}[Z] = np$. Now we fix $\lambda = np$ as constant take a limit on $n \to \infty$, which implies that $p = \frac{\lambda}{n} \to 0^+$. We can show that (Lemma A.1)

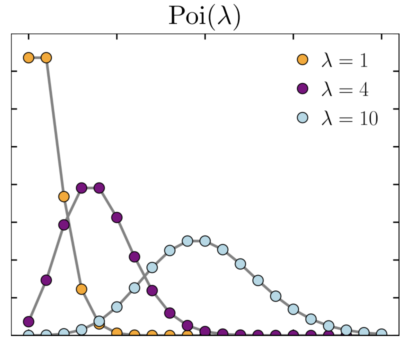
$$\mathrm{Poi}(\lambda)(k) = \lim_{n \to \infty} \mathrm{Bin}\left(n, \frac{\lambda}{n}\right)(k) \tag{2}$$

Since $\mathbf{E}[Z] = np$ and $\mathrm{Var}\,(Z) = np(1-p)$, we can also infer from (2) that $\mathbf{E}[X] = \lambda$ and $\mathrm{Var}\,(X) = \lambda$. Here are some plots from Wikipedia:

$\mathrm{Bin}(n, p)$



$\mathrm{Poi}(\lambda)$



2 Properties

If $X_1 \ldots X_N$ where $X_i \sim \mathrm{Poi}(\lambda_i)$ independently, then (Lemma A.2)

$$\sum_{i=1}^{N} X_i \;\sim\; \mathrm{Poi}\left(\sum_{i=1}^{N} \lambda_i\right) \tag{3}$$

This property can be used to justify a normal approximation of the Poisson variable (which is visually evident in the plot above).

Lemma 2.1. Let $X_\lambda \sim \text{Poi}(\lambda)$. As $\lambda \to \infty$, we have

$$X_\lambda \overset{\text{approx.}}{\sim} \mathcal{N}(\lambda, \lambda) \tag{4}$$

Proof. WLOG we assume λ is a whole number. By (3), we can reparamterize $X_\lambda = \sum_{i=1}^{\lambda} X_i$ where $X_i \sim \text{Poi}(1)$ independently. By the central limit theorem, $\frac{1}{\lambda} X_\lambda \overset{\text{approx.}}{\sim} \mathcal{N}(1, \frac{1}{\lambda})$ or $X_\lambda \overset{\text{approx.}}{\sim} \mathcal{N}(\lambda, \lambda)$ as $\lambda \to \infty$. □

Application 2.2 (Stirling's approximation[1])**.**

$$n! \to \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \text{ as } n \to \infty \tag{5}$$

Proof. By Lemma 2.1,

$$\begin{aligned}
\text{Poi}(n)(k) \approx \mathcal{N}(n, n)(k) \quad &\Leftrightarrow \quad \frac{n^k}{e^n k!} \approx \frac{1}{\sqrt{2\pi n}} e^{\frac{(k-n)^2}{n}} \\
&\Rightarrow \quad \frac{n^n}{e^n n!} \approx \frac{1}{\sqrt{2\pi n}} \qquad \text{(by choosing } k = n\text{)} \\
&\Leftrightarrow \quad n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n
\end{aligned}$$

where the approximation becomes exact as $n \to \infty$. □

Corollary 2.3. $\ln(n!) = n \ln n - n + O(\ln n)$

References

Robbins, H. (1955). A remark on stirling's formula. *The American mathematical monthly*, **62**(1), 26–29.

[1] A non-asymptotic generalization is given by Robbins (1955): for all $n \in \mathbb{N}$,

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$$

For instance, with $n = 10$ we have $1 \leq \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \leq 1.0084$.

A Lemmas

Lemma A.1.

$$\lim_{n\to\infty} \operatorname{Bin}\left(n, \frac{\lambda}{n}\right)(k) = \frac{\lambda^k}{e^\lambda k!}$$

Proof. We have

$$\operatorname{Bin}\left(n, \frac{\lambda}{n}\right)(k) = \binom{n}{k}\frac{\lambda^k}{n^k}\left(1-\frac{\lambda}{n}\right)^{n-k} = \frac{n(n-1)\cdots(n-k+1)}{n^k}\left(1-\frac{\lambda}{n}\right)^{n-k}\frac{\lambda^k}{k!}$$

Thus by the usual property of a limit,

$$\lim_{n\to\infty} \operatorname{Bin}\left(n, \frac{\lambda}{n}\right)(k) = \left(\lim_{n\to\infty}\frac{n(n-1)\cdots(n-k+1)}{n^k}\right)\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n-k}\right)\frac{\lambda^k}{k!}$$

The first limit clearly converges to 1. More formally, distributing the denominator to the k terms in the numerator, and distributing the limit, we have

$$\lim_{n\to\infty}\frac{n(n-1)\cdots(n-k+1)}{n^k} = \lim_{n\to\infty}\left(1-\frac{1}{n^k}\right)\cdots\lim_{n\to\infty}\left(1-\frac{k-1}{n^k}\right) = 1$$

For the second limit, we have

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n-k} = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k} = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n} = \frac{1}{e^\lambda}$$

where the last equality follows from the limit definition of e^λ.[2] □

Lemma A.2. Let $X \sim \operatorname{Poi}(\alpha)$ and $Y \sim \operatorname{Poi}(\beta)$ be independent. Then $X + Y \sim \operatorname{Poi}(\alpha+\beta)$.

Proof. We show that $\Pr(X+Y=k) = \frac{(\alpha+\beta)^k}{e^{\alpha+\beta}k!}$:

$$\begin{aligned}
\Pr(X+Y=k) &= \sum_{i=1}^{k}\Pr(X=i\ \wedge Y=k-i) \\
&= \sum_{i=1}^{k}\Pr(X=i)\Pr(Y=k-i) \\
&= \sum_{i=1}^{k}\left(\frac{\alpha^i}{e^\alpha i!}\right)\left(\frac{\beta^{k-i}}{e^\beta (k-i)!}\right) \\
&= \frac{1}{e^{\alpha+\beta}}\sum_{i=1}^{k}\frac{\alpha^i\beta^{k-i}}{i!(k-i)!} \\
&= \frac{1}{e^{\alpha+\beta}k!}\sum_{i=1}^{k}\frac{k!}{i!(k-i)!}\alpha^i\beta^{k-i} \\
&= \frac{1}{e^{\alpha+\beta}k!}(\alpha+\beta)^k \qquad \text{(binomial theorem)}
\end{aligned}$$

□

[2] It can also be derived directly. Let $u = \ln(1-\lambda/n)$ where as $n\to\infty$, we have $u \to 0^-$. We also have $n = \lambda/(1-e^u)$. Then

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = \lim_{n\to\infty}\exp\left(n\ln\left(1-\frac{\lambda}{n}\right)\right) = \lim_{u\to 0^-}\exp\left(\frac{\lambda u}{1-e^u}\right) = \exp\left(\lambda\left(\lim_{u\to 0^-}\frac{u}{1-e^u}\right)\right) \overset{*}{=} \exp\left(\lambda\left(\lim_{u\to 0^-}\frac{1}{-e^u}\right)\right) = e^{-\lambda}$$

where $\overset{*}{=}$ uses Bernoulli's rule.